The voter who wasn't there: Referenda, Representation and Abstention*

Paulo P. Côrte-Real† Paulo T. Pereira‡

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Abstract

We analyze single binary-choice voting rules and identify the presence of the No-Show paradox in this simple setting, as a consequence of specific turnout or quorum conditions that are included in actual rules. Since these conditions are meant to ensure a representative outcome, we formalize this concern and reach our main result: no voting rule can ensure representation if abstention is possible, unless restrictive assumptions are made on the preference domain of abstainers. We then focus on the main referendum systems and show that appropriate restrictions do make them compatible with representation.

The main purpose of our paper is, however, to provide a tool for referendum design: rather than imposing arbitrary restrictions on the preference domain of non-voters, we recommend instead that a conscious choice be made on how abstention is to be interpreted and that this choice be used to derive the corresponding referendum rule.

JEL Classification: D71

*The idea for this paper started with some jocose but insightful notes written by José João Marques da Silva at the time of the first referendum held in Portugal (1998). When José João passed away in August 2000, ISEG lost a bright, interested and friendly scholar. May we dedicate this paper to his memory. This paper was presented at the 2002 Annual Meeting of the Public Choice Society and Economic Science Association, San Diego, CA and a preliminary version was presented at the 2001 Annual Meeting of the European Public Choice Society, Paris. We would like to thank Mathew Braham, Moshé Machover, Eric Maskin, Vincent Merlin, Hannu Nurmi, Katri Sieberg, Frank Steffen, and two anonymous referees for helpful comments. The usual proviso applies.

†Faculdade de Economia, Universidade Nova de Lisboa, Campus de Campolide, 1099-032 Lisboa, Portugal.

‡Instituto Superior de Economia e Gestão, Universidade Técnica de Lisboa, Rua Miguel Lúpi 20, 1200 Lisboa, Portugal.

1 Introduction

Direct democracy is increasingly popular in European countries[1]. While countries like Switzerland [8], Italy and Ireland have been using direct democracy procedures for decades, and for a great variety of issues, in the last decade we have seen France, Austria, Sweden, Finland [3], Norway, and (repeatedly) Denmark [17] holding referenda on issues related to European Integration - such as the decision to join the Union, the ratification of Treaties and the adoption of the Euro -, as well as Portugal introducing the referendum institution to decide on abortion laws.

Although the use of referenda is widespread, it is far from homogeneous: differences include the right of initiative (group of citizens, Parliament, President or monarch), the object of the question (constitutional amendment, ordinary Act approved by the Parliament, bill proposal, local issue), the intended effects of the referendum (approval or veto), the domain of voters (all electors or electors of particular states) and the rules that ultimately decide the outcome [12]. For the purposes of our paper, we restrict the analysis to the typical referendum, that is held to decide on a unique change to the status quo - and the citizens are called to either ratify or veto that change. From here on, we will simply examine the case where a vote for 'No' is a vote for the status quo whereas a vote for 'Yes' supports the change - and we also study the consequences of this assumption and the extent to which it is costless.

The public and social choice literature has seldom focused on simple Yes-No voting for an equally simple reason: in the presence of just two alternatives - a change versus the status quo -, the Condorcet-Arrow problems of collective choice disequilibrium vanish and most decision-making rules lead to the same outcome. A simple majority rule (associated with some tie-breaking rule) should then be used, ensuring the choice of the Condorcet winner[2]. Nurmi [14], in his analysis of referenda, concludes that every referendum should indeed be restricted to a unique binary decision in order to avoid agenda manipulation, and also that its result should be binding. In turn, the relevance of the simple majority rule in the simple binary decision setting was established by May's [10] well-known characterization: it is the unique well-defined rule that is both anonymous and neutral (i.e. independent of the names of the voters, and of the names of the alternatives, respectively) and that responds positively to changes in the preferences of voters. We find further support for the use of the majority rule in the work

[1] Apart from India, Israel, Japan, the Netherlands and the United States (on a federal level), all major democracies have used national referenda.

[2] The Condorcet winner is the alternative that defeats all others in a pairwise comparison.

of Felsenthal and Machover [6]: defining ternary voting games as a generalization of simple voting games where abstention[3] is an additional option for voters, they conclude that the simple majority rule achieves the highest 'democratic participation index' defined as responsiveness to the desires of the average voter, inherently recommending its use in the presence of abstention.

In an apparent paradox, however, several countries still choose to resort to referenda where that simple majority requirement is coupled with an additional turnout condition - and only if that condition is met does the majority rule apply. Similarly, a quorum requirement is a widespread feature of decision-making in all types of committees, ranging from legislative bodies and general assemblies to simple department meetings. This type of additional requirement is usually said to stem from the concern with representation - the need to ensure that the actual voters are 'good' representatives of the potential voters - and need not be, therefore, a paradox in itself. However, the inclusion of a turnout condition in addition to the majority condition may create a virtual dilemma for supporters of the status quo: if they do not vote, they may contribute for the approval of the issue, because their abstention may lead 'Yes' supporters to fulfill the majority condition; if they vote (or attend the meeting) they may cause the approval of the issue by contributing to fulfill the turnout condition. The latter possibility constitutes a real paradox in the heretofore paradox-free world of two alternatives: voters may be able to manipulate the voting outcome by abstaining.

This possibility of manipulating outcomes through abstention is the definition of the abstention or 'no-show' paradox, a term coined by Fishburn and Brams [7] for a setting where three or more proposals are to be voted upon, and also explored by Pérez [15] and Nurmi[13]. Lepelley and Merlin [9] distinguish two relevant cases in that setting: the positive abstention paradox (PAP) where "some voters with a loser a_j ranked first are deleted (or they abstain) and a_j becomes a winner" and the negative abstention paradox (NAP) where "some voters with a winner a_j ranked last are deleted (or they abstain) and a_j becomes a loser" [9, p.55]. These paradoxes, however, hinge on the existence of a third alternative. In our case, given that there is only a binary choice to be made, the distinction between PAP and NAP vanishes and the paradox is caused by the addition of the turnout condition to the decision rule. Despite this distinction, the essence of the issue is the same and we will therefore apply the same phrase to describe it.

In this paper, we first try to formalize the concern with a representative outcome and we argue that this concern can only be reconciled with a turnout condition in referenda (or a quorum in committees), or indeed

[3] We also use the term 'abstention' defined as simple non-voting.

with any referendum system, through specific interpretations of abstention. Awareness of this connection might be of relevance in referendum design - rather than defining a rule that is meant to ensure a representative outcome, which results in a certain interpretation of abstention, we suggest a reversal of this process: making a deliberate choice on the appropriate interpretation of abstention (namely, avoiding the aforementioned paradox) and deriving the voting rule from that interpretation. We base our analysis on referenda, although our results extend to decision-making in committees as well as to other institutional settings involving a simple binary choice.

In section 2, we introduce the formal framework, present a typology of referenda and identify the type that causes the abstention paradox. In section 3, we formalize the notion of representation and develop an axiomatic study of the model, presenting our main impossibility result: in the absence of a specific interpretation of the preferences of abstainers, no rule that is solely based on the preferences of the set of actual voters can simultaneously satisfy representation and consistency with the application of the Condorcet principle[4] to the population. Designing a referendum that satisfies these axioms therefore requires some interpretation of abstention. In section 4, we discuss some possibilities of interpretations and present specific cases that make existing rules compatible with our axioms. In section 5, we conclude and suggest further research on the use of this tool for referendum design.

2 Single binary-choice referenda: the model and actual rules

We consider the following scenario: a population of potential voters is called upon to vote in a single-issue referendum. We assume that the referendum question is phrased so that a vote for 'Yes' is a vote against the status quo, whereas a vote for 'No' supports the status quo[5]. Each individual has complete preferences over these two alternatives and can take one of three possible actions: vote 'Yes', vote 'No' or abstain (an implicit assumption is that there are no blank or null votes).

Formally, we assume an infinite population of potential agents indexed by the natural numbers (the set $\mathbf{N}$); let $\aleph$ be the class of non-empty finite subsets of $\mathbf{N}$; let $N \in \aleph$ denote the electorate or set of potential voters. Now let the set of actual voters be any $N' \subseteq N$ with $N' \in \aleph$. The set of abstain-

[4]The Condorcet principle defends the selection of the Condorcet winner, whenever that winner exists.

[5]We will derive the consequences of this assumption when we address referendum systems that are not symmetric with regard to the weights given to the status quo and the proposed change. In that case, and under our assumption, the system will not be neutral i.e. the names of the alternatives will be of relevance in the outcome.

ers is then $N \backslash N'$. Each agent $i \in N'$ is endowed with a rational[6] preference relation R_i on the set of outcomes $A = \{Yes, No\}$. Let $\Re$ denote the class of all such preference relations. We denote the preference profile of the set of actual voters by $R_{N'} = (R_i)_{i \in N'} \in \Re^{N'}$. We allow for the possibility of imposing a restriction on the preference domain of the set of abstainers, based on the remaining information and let $D_{(N', N \backslash N', R_{N'})}$ denote that domain, where $D_{(N', N \backslash N', R_{N'})} \subseteq \Re^{N \backslash N'}$. Let $R_{N \backslash N'}$ denote the preference profile of the set of abstainers where $R_{N \backslash N'} = (R_i)_{i \in N \backslash N'} \in D_{(N', N \backslash N', R_{N'})} \subseteq \Re^{N \backslash N'}$. A polity is then an electorate $N \in \aleph$, a set of actual voters $N' \in 2^N \backslash \varnothing$, its associated preference profile $R_{N'} = (R_i)_{i \in N'} \in \Re^{N'}$, and a preference profile $R_{N \backslash N'} \in D_{(N', N \backslash N', R_{N'})} \subseteq \Re^{N \backslash N'}$ for the abstainers. We denote each polity by $(N', N \backslash N', R_{N'}, R_{N \backslash N'})$. For simplicity, whenever $N' = N$, we simply denote the polity by (N, R). A voting rule is a mapping V that associates each polity with a single outcome $V(N', N \backslash N', R_{N'}, R_{N \backslash N'}) \in A$.

Moulin [11], in his study of the abstention paradox, also defines a voting rule as a function of the set of voters and their preferences, rather than their actions. A connection to actual systems would then entail the assumption that if a vote is actually cast, it reflects a preference i.e. if an agent votes, he votes for his most preferred alternative. In order to focus on the strategic problem of participation associated with the abstention paradox, another potential strategic issue - whether voters might have an incentive to vote for their worst alternative rather than their best - is ignored. With the purpose of connecting our preference-based voting rule with the action-based existing systems, we also introduce that behavioral assumption - a vote actually cast reveals a preference -, bearing in mind that the rules we study in this paper, as well as all rules that are actually used in any of the existing systems, are consistent with this assumption from a strategic viewpoint. For simplicity, we also assume that half of the total number of indifferent agents who vote choose Yes.

We can now introduce some additional notation in order to express the basic rules used in most referenda and then proceed with the analysis.

Let $n = |N|$ and $n' = |N'|$. Given our behavioral assumptions, we define

$$\beta = \frac{|i \in N' : Yes P_i No| + \frac{1}{2} |i \in N' : Yes I_i No|}{|N'|}$$

to be the proportion of voters in N' that vote for Yes[7]. In each of the

[6] The implication of rationality in this setting is merely completeness, which does, however, allow for indifference.

[7] The inclusion of $\frac{1}{2} |i \in N' : Yes I_i No|$ in the numerator of β would be problematic if $|i \in N' : Yes I_i No|$ were odd. If that were the case, and the cardinality of the set were $2k+1$, where k is an integer, we would simply have k agents voting for Yes and k agents voting for No. The $(2k+1)th$ agent, that we call i, would vote so that $V(N, N', R_{N'}) =$

current referendum systems, $V(N', N \backslash N', R_{N'}, R_{N \backslash N'}) = Yes$, defeating the status quo, if and only if some subset of the following conditions hold[8]:

1. The Majority (of voters) condition, $\beta > \lambda$. Yes wins only if the proportion of votes for Yes in the set of actual voters exceeds a constant λ.

2. The Majority Threshold condition, $\beta \frac{n'}{n} > \theta$: Yes wins only if the proportion of actual votes for Yes in the electorate exceeds a constant θ.

3. The Voting Threshold condition, $\frac{n'}{n} > \tau$: Yes wins only if the proportion of actual voters in the electorate exceeds a constant τ.

We can then use subsets of these conditions to define the main existing rules:

Definition 1 *Rule $S(\lambda)$ is such that $V(N', N \backslash N', R_{N'}, R_{N \backslash N'}) = Yes$ if and only if $\beta > \lambda$.*

Definition 2 *Rule $MT(\theta)$ is such that $V(N', N \backslash N', R_{N'}, R_{N \backslash N'}) = Yes$ if and only if $\beta > \frac{1}{2}$ and $\beta \frac{n'}{n} > \theta$.*

Definition 3 *Rule $VT(\tau)$ is such that $V(N', N \backslash N', R_{N'}, R_{N \backslash N'}) = Yes$ if and only if $\beta > \frac{1}{2}$ and $\frac{n'}{n} > \tau$.*

Table 1 provides specific examples of these rules.

Insert Table 1 here

We should point out that we assume that a tie, that may occur if either of the conditions imposed by each system is met with equality, is broken lexicographically and No wins. This assumption has no effect on the main results and we therefore keep it throughout the analysis.

All systems require condition 1. The simple referendum, S, requires no additional condition. This is the case of Ireland where constitutional amendments have to be approved in a referendum, but where no minimal participation rate is set. In Switzerland, 50000 citizens may require a referendum on a normal act of parliament, and a majority of votes will suffice to approve it. With no threshold condition, there is perfect symmetry between Yes and No. The Namibian referendum for constitutional amendments, however, requires a majority of $\frac{2}{3}$ and symmetry is therefore lost.

Other systems require an additional turnout condition for the result to be binding. Since we defined Yes as a vote for a change and No as a vote for the status quo, the failure to satisfy this condition implies that no change directly

$V(N, N' \backslash \{i\}, R_{N' \backslash \{i\}})$. Since the relevant assumption is that indifferent agents are not decisive, from now on we work as if $|i \in N' : Yes I_i No|$ were always even.

[8] We omit the case of referenda in federations, where a different additional condition applies: the majority of states/cantons.

results from the referendum and therefore leads to the ultimate choice of No by the voting rule. The names of the alternatives become therefore relevant and, for Yes to effectively win, the turnout condition must be met - and the usual underlying rationale for this type of condition is based on the desire to ensure that the result is representative of the will of the electorate as a whole. The turnout condition can be formalized as either condition 2 or condition 3. A quick analysis of the table is enough to conclude that the VT referendum, that pairs Majority with Voting Threshold, implies the existence of an effective majority threshold and that the MT referendum, that pairs Majority with Majority Threshold, also implies the existence of a voting threshold. However, there is no equivalence between the two types of referenda. Looking at the Italian case, the conditions for the approval of Yes do imply the exact approval conditions in the German case. Yet, when Germany establishes that in each Land, the majority that supports a redesign of Lander boundaries has to represent over $\frac{1}{4}$ of the electorate, the only immediate consequence is that the voting threshold in that Land must also be at least $\frac{1}{4}$. The requirements of the VT referendum are therefore stronger (Figure 1).

Insert Fig 1 here

The No-Show Paradox arises when an agent is better off abstaining than voting for his preferred choice. The VT referendum generates this paradox: if agent i is a No supporter and $\beta > \frac{1}{2}$, $\frac{n'}{n} > \frac{1}{2}$ but $\frac{n'-1}{n} \leq \frac{1}{2}$, it is optimal for i to abstain instead. The S and MT referenda do not cause this problem: in both systems, β will increase if a No supporter decides instead to abstain, and in MT, $\beta\frac{n'}{n}$ remains unaltered by the additional abstention of a No supporter. The Italian and Portuguese referenda, by requiring participation, may actually discourage it. We formalize this notion of participation in our next section where we conduct an axiomatic study of this model.

3 Condorcet-Consistency, Representation and Abstention

The Condorcet principle defends the selection of the Condorcet winner - an alternative that defeats all others in a pairwise comparison - whenever that winner exists. Moulin [11] shows that, although the Condorcet principle implies the No-Show Paradox among four or more candidates, this need not be the case for three candidates or less. Namely, for a single binary-choice referendum, a simple majority of voters together with some tie-breaking condition would be consistent with the Condorcet principle and would not cause the Paradox. However, Moulin makes no distinction between the set of actual voters and the whole electorate, in a situation where those two sets might not coincide. Our definition is explicitly based on all possible sets of actual voters.

Let $n_{YN}(N', R_{N'}) = |i \in N' : YesP_iNo| - |i \in N' : NoP_iYes|$ denote the balance of the number of voters strictly preferring Yes to No[9]. Yes is a Condorcet winner if and only if $n_{YN}(N', R_{N'}) > 0$. No is a Condorcet winner if and only if $n_{YN}(N', R_{N'}) \leq 0$.

Definition 4 *V satisfies **Condorcet Consistency** if, for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$ and for all $R_{N \backslash N'} \in D_{(N', N \backslash N', R_{N'})} \subseteq \Re^{N \backslash N'}$, $n_{YN}(N', R_{N'}) > 0 \Leftrightarrow V(N', N \backslash N', R_{N'}, R_{N \backslash N'}) = Yes$.*

This definition holds for all possible sets of actual voters where one of those sets is the whole electorate. However, rather than the rule satisfying Condorcet Consistency, it might instead be deemed desirable to apply that principle solely to the electorate as a whole: should the population of potential voters actually vote, the majority would choose the outcome. This is, of course, a weaker requirement but will be useful once we introduce a notion of representation.

Definition 5 *V satisfies **Population Condorcet Consistency** if, for all $N \in \aleph$, for all $R \in \Re^N$, $n_{YN}(N, R_N) > 0 \Leftrightarrow V(N, R) = Yes$.*

Now, and so as to rule out the Abstention Paradox, we follow Moulin and introduce the Participation axiom, that requires that no agent strictly benefit from abstaining rather than voting for his most preferred alternative:

Definition 6 *V satisfies **Participation** if, for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$, for all $R_{N \backslash N'} \in D_{(N', N \backslash N', R_{N'})} \subseteq \Re^{N \backslash N'}$, and for all $i \in N'$, $V(N', N \backslash N', R_{N'}, R_{N \backslash N'}) R_i V(N' \backslash \{i\}, N \backslash (N' \backslash \{i\}), R_{N' \backslash \{i\}}, R_{N \backslash (N' \backslash \{i\})})$.*

Remark 1 *We have equivalence between $n_{YN}(N', R_{N'}) > 0$ and $\beta > \frac{1}{2}$. Therefore, the only rule that satisfies Condorcet-Consistency is $S(\frac{1}{2})$. Given that $S(\frac{1}{2})$ also satisfies Participation, our result is consistent with Moulin's (up to the tie-breaking assumption).*

At this point, we need to address additional issues that arise when the sets of potential and actual voters differ.

The first issue is the domain for preferences of abstainers. Given that these agents do not express their preferences through their votes, we may not wish to make any restrictions on those preferences, based either on the preferences of agents who do vote or on the information about the sets of

[9] Alternatively, we could have defined $n_{YN}(N, R) = |i \in N : Yes P_i No| + \frac{1}{2} |i \in N : Yes I_i No| - (|i \in N : No P_i Yes| + \frac{1}{2} |i \in N : Yes I_i No|)$ if $|i \in N : Yes I_i No|$ is even. Once more, if the cardinality of the set of indifferent agents were odd and equal to $2k+1$, we would include the additional $(2k+1)th$ agent so that the sign of n_{YN} remains unchanged.

potential and actual voters. In that case, we should apply the following axiom:

Definition 7 *If for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, and for all $R_{N'} \in \Re^{N'}$, $D_{(N', N\backslash N', R_{N'})} = \Re^{N\backslash N'}$, there is an* ***unrestricted preference domain for the set of abstainers****.*

Besides the domain for the preferences of abstainers, there is the additional matter of whether those preferences should have a direct effect on the voting outcome. Although we initially allowed the voting rule to select either 'Yes' or 'No' according to all the information that characterizes a polity, it might be deemed desirable to exclude the preferences of abstainers from the information used by a voting rule. Indeed, no actual system explicitly depends on that piece of information - voting rules are defined as a function of the actions of the actual voters, the size of the electorate and the size of the set of actual voters. This is also the reason why, when we adapted our model to represent actual systems, we were able to bypass the debate on a positive theory of turnout[10] and place no *a priori* restriction on the preferences of the abstaining segment of the electorate.

With the purpose of ruling out the effect of 'irrelevant' potential voters, we should then introduce an additional invariance axiom: regardless of the preferences of individuals who do not vote, the voting rule should produce the same outcome.

Definition 8 *V satisfies* ***Independence of Preferences of Abstainers*** *if, for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$ and for all $R_{N\backslash N'}, R'_{N\backslash N'} \in D_{(N', N\backslash N', R_{N'})} \subseteq \Re^{N\backslash N'}$, $V(N', N\backslash N', R_{N'}, R_{N\backslash N'}) = V(N', N\backslash N', R_{N'}, R'_{N\backslash N'})$.*

The next step in the analysis is to introduce an axiom that formalizes the concern with a representative outcome. Although this concern is usually expressed through the inclusion of a turnout condition in referenda, the underlying problem is not the number of agents who vote but rather whether the outcome coincides with the outcome that would result if the whole electorate were to reveal their preferences through voting.

If we want the set of actual voters to accurately represent the electorate, we then need to ensure that the rule is such that, regardless of whether there is agreement between a majority of voters and a majority of the electorate as a whole, the outcome is the one that would occur were that whole electorate to vote. We therefore need the following axiom and its underlying notion of juxtaposition invariance:

[10] Downs [5] presented the seminal model of turnout, an issue more recently discussed, among many others, by Aldrich [1], Young [18] and Sen [16].

Definition 9 *V satisfies* ***Representation*** *if for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$ and for all $R_{N\backslash N'} \in D_{(N', N\backslash N', R_{N'})} \subseteq \Re^{N\backslash N'}$, $V(N, R) = V(N', N\backslash N', R_{N'}, R_{N\backslash N'})$.*

Our notion of representation[11] combined with the unrestricted preference domain for the set of abstainers requires that the preferences of the agents who vote be such that their votes suffice to ascertain the will of the electorate as a whole; combining this with the Condorcet principle applied to the electorate would therefore ensure that the procedure is democratic. This is then a preference-based notion of democracy - democracy means that the alternative that the whole electorate prefers is chosen -, distinct from the (potential-)action-based notion presented by Felsenthal and Machover [6].

However, this notion of democracy is too demanding and we reach the following results:

Proposition 1 *With the unrestricted preference domain for the set of abstainers, V satisfies Representation and Independence of Preferences of Abstainers if and only if V is such that for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$, for all $R_{N\backslash N'} \in \Re^{N\backslash N'}$, and for all $R' \in \Re^N$, $V(N', N\backslash N', R_{N'}, R_{N\backslash N'}) = V(N, R)$ and $V(N, R) = V(N, R')$.*

Proof. Sufficiency is straightforward. Representation alone leads to the necessity of the first equality. Now towards a contradiction, assume that $N \in \aleph$ and $R, R' \in \Re^N$ are such that $V(N, R) \neq V(N, R')$. Let N' and $N\backslash N'$ be an arbitrary partition of N where $N' \neq \varnothing$ and $N' \subset N$. By Representation, $V(N', N\backslash N', R_{N'}, R_{N\backslash N'}) = V(N, R)$ and $V(N\backslash N', N', R'_{N\backslash N'}, R'_{N'}) = V(N, R')$. Let $R'' \in \Re^N$ be such that $R''_i = R_i$ for all $i \in N'$ and $R''_i = R'_i$ for all $i \in N\backslash N'$. Again by Representation, $V(N', N\backslash N', R''_{N'}, R''_{N\backslash N'}) = V(N\backslash N', N', R''_{N\backslash N'}, R''_{N'}) = V(N, R'')$. But by Independence of Preferences of Abstainers, $V(N', N\backslash N', R''_{N'}, R''_{N\backslash N'}) = V(N', N\backslash N', R_{N'}, R_{N\backslash N'})$ and $V(N\backslash N', N', R''_{N\backslash N'}, R''_{N'}) = V(N\backslash N', N', R'_{N\backslash N'}, R'_{N'})$ and we reach a contradiction. ■

Theorem *With the unrestricted preference domain for the set of abstainers, there is no rule that simultaneously satisfies Representation, Population Condorcet Consistency and Independence of Preferences of Abstainers.*

[11] A weaker notion would be a Reinforcement-type axiom requiring that, when the majorities of the sets of potential and actual voters agree, the same choice be made by the voting rule. In that case, and out of the main set of existing rules, $S(\frac{1}{2})$ would be the only one that satisfied that notion. It is easy to check, however, that $S(\frac{1}{2})$ does not satisfy Representation, resulting in an incompatibility between the latter and Condorcet Consistency.

Proof. Let $N \in \aleph$, and $R, R' \in \Re^N$ be such that $n_{YN}(N,R) > 0$ and $n_{YN}(N,R') \leq 0$. Then, by Proposition 1, $V(N,R) = V(N,R')$, violating Population Condorcet Consistency. ■

The first result allows us to write $V(N', N \backslash N', R_{N'}, R_{N \backslash N'})$ as simply $V(N)$, provided that V satisfies Representation and Independence of Preferences of Abstainers, and that the preference domain of abstainers is unrestricted. This leads, in turn, to the impossibility result in the Theorem[12]. This result is very intuitive: we are looking for a voting rule that on the one hand cannot depend on the preferences of abstainers but on the other hand must reflect the wishes of the majority of the whole electorate, whatever the preferences of the segment who abstain may be.

It is now clear that the systems described at the end of Section 2 may only satisfy Representation, Population Condorcet Consistency and Independence of Preferences of Abstainers if appropriate restrictions on the preference domain of abstainers are introduced. Therefore, if the goal is to implement the will of the electorate as a whole and to ensure that the result of a referendum is the same for whatever preferences abstainers may have within their preference domain, that domain must be restricted[13]. Whether this restriction is based on any of the available information on the preferences of actual voters and on the sets of potential and actual voters, this means that preferences of abstainers that are not disclosed by the voting process are, or must be, nevertheless, interpreted. In our next section, we clarify this idea and show that most of the existing systems may indeed satisfy the three axioms, under specific interpretations of abstention.

4 Interpreting Abstention

We are now aware that when any referendum system, defined through a voting rule, intends to ensure an accurate representation of the preferences of the electorate, assumptions on the preferences of abstainers are inherently made. We propose instead that the process be reversed: that those assumptions be consciously decided upon and then used as a means to define the referendum rule.

In this section, we therefore invert the usual process and show that for appropriate restrictions on the preferences of the abstaining agents, most of

[12] Adding the Participation requirement, the equivalence in Proposition 1 would still hold. However, even though Participation also produces no conflict with Population Condorcet Consistency (or indeed with Condorcet Consistency, as Moulin [11] states for our binary decision case), adding Participation to the Theorem would still naturally yield the impossibility that results from the incompatibility of the original axioms.

[13] In this sense, our result bears a resemblance to Arrow's [2] seminal result.

the referendum rules that are actually applied may satisfy Representation, Population Condorcet Consistency and Independence of Preferences of Abstainers - and actually, that a restriction on the proportion of abstainers that favor Yes will suffice. This type of restriction amounts to a specific interpretation of abstention, but although we redirect the problem of defining a referendum system to the problem of interpreting abstention, we do not intend to provide any recommendation about how to solve the latter.

One possible view on abstention is that agents who abstain are indifferent towards the issue, given that they had the same prior power (to vote) and decided not to exercise it; in that case, those agents are dummies and can be ignored. Another possible interpretation is specific to referenda: since referenda are held to decide on a change to the status quo, people who abstain do so because they are comfortable with the status quo and abstention is in that case interpreted as support for No[14]. A legal analogy might be useful in understanding these cases: whereas in the latter, the burden of proof is on Yes, in the former, both Yes and No are innocent until proven guilty. Each of these views depends however on a specific positive theory of turnout and on specific assumptions about independence or homogeneity of the preferences of the electorate. Like Braham and Steffen [4], we leave possible positive (and normative) statements about this issue to future research. Instead, we merely consider the main types of referenda and offer possible interpretations that could justify each of them as representative of the wishes of the electorate.

We denote the proportion of abstainers[15] in $N \backslash N'$ that favor Yes by

$$\alpha = \frac{|i \in N \backslash N' : Yes P_i No| + \frac{1}{2}|i \in N \backslash N' : Yes I_i No|}{|N \backslash N'|}.$$

Since N and N' are finite sets with integer cardinalities, we can only have $\alpha \in \left\{0, \frac{1}{n-n'}, \frac{2}{n-n'}, ..., 1\right\}$. In order to simplify the exposition, however, we use $\alpha \in [0,1]$. Still, for each α we consider in this section, there is an appropriate α' that may be written as $\frac{k}{n-n'}$ for some $k = 0, 1, ..., n - n'$ such that $|\alpha' - \alpha| \leq \frac{1}{2(n-n')}$, yielding the same conclusions.

There are three main axioms we are interested in. All the rules that we study satisfy Independence of Preferences of Abstainers. We can now apply this additional notation in order to condense the other two requirements of Representation and Population Condorcet Consistency.

[14]...or a vote against holding the referendum. This must, however, be specific to the referendum scenario because otherwise, in a general election, abstention would also have to be interpreted as votes for the incumbent party - or, alternatively, against democracy itself.

[15]In this section, we always assume that the set of abstainers $N \backslash N'$ is non-empty.

Remark 2 *V satisfies Representation and Population Condorcet Consistency if for all* $N \in \aleph$, *for all* $N' \in 2^N \backslash \varnothing$, *for all* $R_{N'} \in \Re^{N'}$ *and for all* $R_{N \backslash N'} \in D_{(N', N \backslash N', R_{N'})} \subset \Re^{N \backslash N'}$, $V(N', N \backslash N', R_{N'}, R_{N \backslash N'}) = Yes$ *if and only if* $\beta n' + \alpha(n - n') > (1-\beta)n' + (1-\alpha)(n-n') \Leftrightarrow \alpha > \left(\frac{n}{2} - \beta n'\right)/(n-n')$.

We begin by analyzing $S(\lambda)$. It is clear that for the Namibian case or for any $S(\lambda)$ such that $\lambda \neq \frac{1}{2}$, these axioms are not compatible since Population Condorcet Consistency alone will fail if the whole electorate votes and β is a value between λ and $\frac{1}{2}$. However, for $\lambda = \frac{1}{2}$, there are several possible values α may take such that S satisfies the three axioms. In fact, there are several possible functions $\alpha(\beta)$ such that for any combination of n and n', this result holds (Fig. 2).

Insert Fig 2 here

Claim 1.1 If for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$, $D_{(N', N \backslash N', R_{N'})}$ is such that for all $R_{N \backslash N'} \in D_{(N', N \backslash N', R_{N'})}$ we have $\alpha = \frac{1}{2}$, then $S(\frac{1}{2})$ satisfies Representation, Population Condorcet Consistency and Independence of Preferences of Abstainers. Furthermore, $S(\frac{1}{2})$ satisfies Condorcet Consistency.

Proof. Using Remark 2, it is straightforward that if $\alpha = \frac{1}{2}$, then $\beta > \frac{1}{2}$ is equivalent to $\beta n' + \alpha(n - n') > (1 - \beta)n' + (1 - \alpha)(n - n')$. ■

As we mentioned above, one possible interpretation of abstention is that all agents who abstain are indifferent towards the issue[16]. This is the case considered in Claim 1.1, where the preferences of the electorate are then restricted to be an additive extension of the preferences of the actual voters. Another possibility is that the electorate is restricted to be a multiplicative extension of the set of voters and that the preferences of the set of voters parallel the preferences of the set of abstainers. This is the case considered in the following Claim:

Claim 1.2 If for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$, $D_{(N', N \backslash N', R_{N'})}$ is such that for all $R_{N \backslash N'} \in D_{(N', N \backslash N', R_{N'})}$ we have $\alpha = \beta$, then $S(\frac{1}{2})$ satisfies Representation, Population Condorcet Consistency and Independence of Preferences of Abstainers. Furthermore, $S(\frac{1}{2})$ satisfies Condorcet Consistency.

Proof. Using Remark 2, it is straightforward that if $\alpha = \beta$, then $\beta > \frac{1}{2}$ is equivalent to $\beta n' + \alpha(n - n') > (1 - \beta)n' + (1 - \alpha)(n - n')$. ■

[16] Given our behavioral assumptions, this would also be the relevant case, should the concern be a notion of average (rather than exact) representation where equal probabilities are assigned to each (unrestricted) preference profile of abstainers.

An analysis of Figure 2 allows us to conclude that in fact any combination of the additive and multiplicative extensions would result in a restriction that makes $S(\frac{1}{2})$ compatible with the three axioms.

The $MT(\theta)$ system, however, fails to allow for the aforementioned additive and multiplicative extensions. Instead, the additional majority threshold condition allows for the following restriction:

Claim 2.1 If for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$, $D_{(N',N\backslash N',R_{N'})}$ is such that for all $R_{N\backslash N'} \in D_{(N',N\backslash N',R_{N'})}$ we have

$$\alpha = \left\{ \begin{array}{c} \frac{1}{2} \text{ if } \beta \leq \frac{1}{2} \\ \min\left\{ \left[(1-2\theta)n\right] / \left[2(n-n')\right], \frac{1}{2} \right\} \text{ if } \beta > \frac{1}{2} \end{array} \right. ,$$

then $MT(\theta)$ with $\theta \leq \frac{1}{2}$ satisfies Representation, Population Condorcet Consistency and Independence of Preferences of Abstainers.

Proof. Imposing $\theta \leq \frac{1}{2}$ ensures that $\alpha \in [0,1]$.

Using Remark 2, we simply need to show that if $\alpha = \frac{1}{2}$ for $\beta \leq \frac{1}{2}$ and $\alpha = \min\left\{ \left[(1-2\theta)n\right] / \left[2(n-n')\right], \frac{1}{2} \right\}$ for $\beta > \frac{1}{2}$, then $\beta > \frac{1}{2}$ and $\beta\frac{n'}{n} > \theta$ are equivalent to $\beta n' + \alpha(n-n') > (1-\beta)n' + (1-\alpha)(n-n')$. If $\beta \leq \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\beta n' + \alpha(n-n') \leq (1-\beta)n' + (1-\alpha)(n-n')$. If $\beta > \frac{1}{2}$ and $\left[(1-2\theta)n\right] / \left[2(n-n')\right] \geq \frac{1}{2} \Leftrightarrow \frac{n'}{2n} \geq \theta$, then $\alpha = \frac{1}{2}$ and $\beta n' + \alpha(n-n') > (1-\beta)n' + (1-\alpha)(n-n')$. If $\beta > \frac{1}{2}$ and $\left[(1-2\theta)n\right] / \left[2(n-n')\right] < \frac{1}{2} \Leftrightarrow \frac{n'}{2n} < \theta$, then $\alpha = \left[(1-2\theta)n\right] / \left[2(n-n')\right]$ and $\beta n' + \alpha(n-n') > (1-\beta)n' + (1-\alpha)(n-n')$ if and only if $\beta\frac{n'}{n} > \theta$. The result then follows because $\beta > \frac{1}{2}$ and $\frac{n'}{2n} \geq \theta$ imply $\beta\frac{n'}{n} > \theta$. ■

Claim 2.1 emphasizes the asymmetry that is introduced to favor the status quo. For $MT(1/4)$, the case applied to each German Land in the Lander redesign question (Figure 3), we can then have that $\alpha = 1/2$ for $\beta \leq 1/2$, and $\alpha = \min\left\{ \frac{n}{4(n-n')}, \frac{1}{2} \right\}$ for $\beta > 1/2$, is an appropriate restriction. In fact, for $\beta > 1/2$, and $\frac{n}{4(n-n')} \leq \frac{1}{2}$, $\alpha = \frac{n}{4(n-n')}$ is the only restriction that can be expressed as function of n and n' that is constant with respect to β. We could then also have the following alternative restriction:

Insert Fig 3 here

Claim 2.2 If for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$, $D_{(N',N\backslash N',R_{N'})}$ is such that for all $R_{N\backslash N'} \in D_{(N',N\backslash N',R_{N'})}$ we have

$$\alpha = \min\left\{ \frac{(1-2\theta)n}{2(n-n')}, \frac{1}{2} \right\}$$

then $MT(\theta)$ with $\theta \leq \frac{1}{2}$ satisfies Representation, Population Condorcet Consistency and Independence of Preferences of Abstainers.

Proof. Except for the case $\beta \leq \frac{1}{2}$,where we now have $\alpha \leq \frac{1}{2}$ and $\beta n' + \alpha(n-n') \leq (1-\beta)n' + (1-\alpha)(n-n')$, we can apply the proof for Claim 2.1. ■

Claim 2.2 parallels Claim 1.1; however, instead of the homogeneity assumption that led to the additive extension interpretation for S, $MT(1/4)$ only allows for a fixed proportion of less than half of the abstainers favoring Yes, whenever $\frac{n}{4(n-n')} \leq \frac{1}{2}$ (Figure 4). In this case, the proportion of abstainers that favor Yes is then increasing in n' for the same n, and decreasing in n for the same n' (and decreasing with respect to the number of abstainers). The assumption on α is therefore favorable to the status quo for a small n', but becomes gradually more neutral as the sample of actual voters becomes larger and, consequently, more informative.

Insert Fig 4 here

Claim 2.3 If for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$, $D_{(N', N\backslash N', R_{N'})}$ is such that for all $R_{N\backslash N'} \in D_{(N', N\backslash N', R_{N'})}$ we have

$$\alpha = \min\left\{\frac{1-2\theta}{2\theta}.\frac{n'}{n-n'}, 1\right\}.\beta$$

then $MT(\theta)$ with $0 < \theta \leq \frac{1}{2}$ satisfies Representation, Population Condorcet Consistency and Independence of Preferences of Abstainers.

Proof. Once more, imposing $0 < \theta \leq \frac{1}{2}$ ensures that $\alpha \in [0,1]$.

Using Remark 2, we simply need to show that if $\alpha = \min\left\{\frac{1-2\theta}{2\theta}.\frac{n'}{n-n'}, 1\right\}.\beta$, then $\beta > \frac{1}{2}$ and $\beta\frac{n'}{n} > \theta$ are equivalent to $\beta n' + \alpha(n-n') > (1-\beta)n' + (1-\alpha)(n-n')$. If $\beta \leq \frac{1}{2}$, $\alpha \leq \frac{1}{2}$ and $\beta n' + \alpha(n-n') \leq (1-\beta)n' + (1-\alpha)(n-n')$. If $\beta > \frac{1}{2}$ and $[(1-2\theta)n'] / [2\theta(n-n')] \geq 1 \Leftrightarrow \frac{n'}{2n} \geq \theta$, then $\alpha = \beta$ and $\beta n' + \alpha(n-n') > (1-\beta)n' + (1-\alpha)(n-n')$. If $\beta > \frac{1}{2}$ and $[(1-2\theta)n'] / [2\theta(n-n')] < 1 \Leftrightarrow \frac{n'}{2n} < \theta$, then $\alpha = [(1-2\theta)n'] / [2\theta(n-n')].\beta$ and $\beta n' + \alpha(n-n') > (1-\beta)n' + (1-\alpha)(n-n')$ if and only if $\beta\frac{n'}{n} > \theta$. The result then follows because $\beta > \frac{1}{2}$ and $\frac{n'}{2n} \geq \theta$ imply $\beta\frac{n'}{n} > \theta$. ■

Claim 2.3 parallels Claim 1.2 insofar as it tells us we could also have the proportion of Yes supporters from the set of abstainers depending positively on the homologous proportion from the set of actual voters. However, α will equal β only when $n' \geq 2\theta n$. Once more for $MT(1/4)$ and the case $\frac{n}{4(n-n')} \leq \frac{1}{2}$ (Figure 4), we have instead that the ratio between α and β is smaller than 1, and again increasing in n' for the same n and decreasing in n for the same n'. For a small n', the proportion of Yes supporters in the set of actual voters is then barely reflected on the assumption on preferences of the abstainers - who are mainly taken to support the status quo. Gradual increments in n' increase the informational power of the sample of actual voters with respect to the electorate, providing a possible justification for having α gradually mirror β. Both Claims 2.2 and 2.3 therefore present restrictions that are similar to the ones proposed for $S(1/2)$ but where the greater weight that is attributed to the status quo is a consequence of more conservative assumptions on α - that allow, nonetheless, for continual adjustments concurrent with the dimension of the sample of actual voters.

For the $VT(\tau)$ system, we again have that neither the additive nor the multiplicative extensions would be associated with appropriate restrictions. Instead, the additional voting threshold condition allows for the following restriction:

Claim 3.1 If for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$, $D_{(N', N\backslash N', R_{N'})}$ is such that for all $R_{N\backslash N'} \in D_{(N', N\backslash N', R_{N'})}$ we have

$$\alpha = \begin{cases} \frac{1}{2} \text{ if } \beta \leq \frac{1}{2} \\ (1 - 2\tau\beta) \,/\, [2(1-\tau)] \text{ if } \beta > \frac{1}{2} \end{cases},$$

then $VT(\tau)$ with $\tau \leq \frac{1}{2}$ satisfies Representation and Population Condorcet Consistency.

Proof. Once more, imposing $\tau \leq \frac{1}{2}$ ensures that $\alpha \in [0,1]$. Using Remark 2, we simply need to show that if $\alpha = \frac{1}{2}$ for $\beta \leq \frac{1}{2}$ and $\alpha = (1 - 2\tau\beta) \,/\, [2(1-\tau)]$ for $\beta > \frac{1}{2}$, then $\beta > \frac{1}{2}$ and $\frac{n'}{n} > \tau$ are equivalent to $\beta n' + \alpha(n - n') > (1-\beta)n' + (1-\alpha)(n-n')$. If $\beta \leq \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\beta n' + \alpha(n-n') \leq (1-\beta)n' + (1-\alpha)(n-n')$. If $\beta > \frac{1}{2}$, $\alpha = (1 - 2\tau\beta) \,/\, [2(1-\tau)]$ and $\beta n' + \alpha(n-n') > (1-\beta)n' + (1-\alpha)(n-n')$ if and only if $\frac{n'}{n} > \tau$. ■

For the Italian or the Portuguese referenda (or for decision-making in committees with a quorum requirement of 50%.), depicted in Figures 5 and 6, we then have that $\alpha = \frac{1}{2}$ if $\beta \leq \frac{1}{2}$, and $\alpha = 1-\beta$ if $\beta > \frac{1}{2}$, is an appropriate restriction; in fact, for $\beta > \frac{1}{2}$, $\alpha = 1-\beta$ is the only function of β that would satisfy the conditions for all n, n'. This type of referendum therefore also imposes an asymmetry between Yes and No, attributing a greater weight to the status quo. In this case, we cannot, however, find any reasonable argument for having the proportion of abstainers favorable to Yes depend negatively on the proportion of voters with the same preferences.

Insert Fig 5 and Fig 6 here

Claim 3.2 If for all $N \in \aleph$, for all $N' \in 2^N \backslash \varnothing$, for all $R_{N'} \in \Re^{N'}$, $D_{(N', N\backslash N', R_{N'})}$ is such that for all $R_{N\backslash N'} \in D_{(N', N\backslash N', R_{N'})}$ we have

$$\alpha = \begin{cases} 0 \text{ if } \frac{n'}{n} \leq \tau \\ \frac{1}{2} \text{ if } \frac{n'}{n} > \tau \end{cases},$$

then $VT(\tau)$ with $\tau \leq \frac{1}{2}$ satisfies Representation and Population Condorcet Consistency.

Proof. Once more, imposing $\tau \leq \frac{1}{2}$ ensures that $\alpha \in [0,1]$. Using Remark 2, we simply need to show that if $\alpha = 0$ for $\frac{n'}{n} \leq \tau$ and $\alpha = 1/2$ for $\frac{n'}{n} > \tau$, then $\beta > \frac{1}{2}$ and $\frac{n'}{n} > \tau$ are equivalent to $\beta n' + \alpha(n-n') > (1-\beta)n' + (1-\alpha)(n-n')$. If $\frac{n'}{n} \leq \tau \leq \frac{1}{2}$, $\alpha = 0$ and, since $\beta \leq 1$, $\beta n' + \alpha(n-n') \leq (1-\beta)n' + (1-\alpha)(n-n')$. If $\frac{n'}{n} > \tau$, $\alpha = 1/2$ and $\beta n' + \alpha(n-n') > (1-\beta)n' + (1-\alpha)(n-n')$ if and only if $\beta > \frac{1}{2}$. ■

If, for $VT(1/2)$, we were to look instead for appropriate restrictions on α that are independent of β (similar to those in Claims 1.1 and 2.2), we would necessarily have $\alpha = 0$ for $n' = n/2$ and $\alpha = 1/2$ for $n' > n/2$, a jump for which we also cannot provide any theoretical explanation. Moreover, in terms of the participation requirement[17], we have already seen that this system is problematic[18].

Comparing the MT and VT systems, the European Commission for Democracy through Law (Venice Commission), in its Guidelines for Constitutional Referendums at National Level adopted in 2001, recommends the use of the former, wherever a turnout condition is deemed to be necessary. The concerns we express regarding the VT system would thus seem to support and provide theoretical grounds for that recommendation.

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[17]Although we only presented one possible restriction on α that would make each system compatible with our axioms, since all such restrictions are equivalent in terms of decision, we can check whether each of these systems satisfies Participation by simply analyzing the one restriction we used.

Denoting by $\alpha(\beta, n, n')$ the restriction on the preferences of the set of abstainers, we encompass all of the aforementioned cases. Participation is then satisfied if for all (β, n, n'), $\alpha = \alpha(\beta, n, n')$ and $\alpha' = \alpha(\beta', n, n'+1)$ are such that neither of the following hold:

1. With $\beta' = \beta n'/(n'+1)$, both $\beta n' + \alpha(n-n') \leq (1-\beta)n' + (1-\alpha)(n-n')$ and $\beta' n' + \alpha'(n-n'-1) > (1-\beta')n' + (1-\alpha')(n-n'-1)$;
2. With $\beta' = \beta(n'+1)/n'$, both $\beta n' + \alpha(n-n') > (1-\beta)n' + (1-\alpha)(n-n')$ and $\beta'(n'+1) + \alpha'(n-n'-1) \leq (1-\beta')(n'+1) + (1-\alpha')(n-n'-1)$.

The first condition represents the case where a No supporter decides to vote, changing the result to Yes and the second condition represents the symmetric case. It is straightforward to check, using the appropriate $\alpha(\beta, n, n')$ that these conditions are never possible for both $S(\frac{1}{2})$ and $MT(\theta)$ with $\theta \leq \frac{1}{2}$ but that although $VT(\tau)$ with $\tau \leq \frac{1}{2}$ never satisfies the second condition, there are indeed cases where the first one holds: just let $\beta > \frac{n'+1}{2n'}$ and let $2n' \leq n < 2(n'+1)$.

[18]It should be noted, in any case, that there might be an endogeneity factor at work here: if participation of No supporters is discouraged, there might be a reason to believe that the preferences of abstainers are not homogeneous.

All of the main referendum systems may be interpreted as attempts to effectively represent the will of the whole electorate. Given that only a fraction of that electorate votes, the outcome of the referendum will only be representative if the preferences of the non-voters are restricted. Awareness of this implication seems of relevance for referendum design: rather than starting by deciding whether or not to impose *ad hoc* turnout conditions, where either decision necessarily entails specific interpretations of abstention, a conscious decision on an adequate interpretation of abstention should be made instead, leading to the definition of the voting rule. We therefore offer the interpretation of abstention as a tool for the design of a representative referendum.

5 Conclusion

We analyze single binary-choice voting rules and identify the presence of the No-Show paradox in this simple setting, as a consequence of specific turnout and quorum conditions that are included, respectively, in actual referendum systems and in rules for decision-making in committees. Given that these conditions are meant to ensure a representative outcome, we try to formalize this concern with representation in the presence of abstention and reach our main impossibility result: no voting rule can ensure an accurate representation of the wishes of the electorate if no restrictive assumptions are made on the preferences of abstainers. A direct consequence of this result is that representation may be achieved for appropriate restrictions on the preference domain of the set of abstainers. We then proceed with the analysis of the main referendum systems and show that such restrictions do exist for most of those systems, although not all of them are necessarily reasonable.

The main purpose of our paper is, however, to provide a tool for referendum design. The definition of a voting system that intends to be representative inevitably results in restrictions on the preferences of non-voters. Unawareness of this implication may lead to the imposition of restrictions that do not satisfy any type of criteria but cause such problems as the No-Show paradox: abstention may be encouraged by how a system chooses to interpret that abstention. We therefore suggest making a conscious choice on how abstention is to be interpreted and using that choice as a tool to derive the corresponding referendum rule.

We do not, however, provide criteria for these choices. They may be based on a positive theory of turnout that may in turn depend on prior assumptions on the preferences of the potential voters, as well as on the distribution of pure costs and benefits of voting among the population; we might also have a decision-maker performing a Bayesian update of the proportion of abstainers who support the status quo based on the information provided by the sample of voters, and on some prior that could reflect a positive degree of risk aversion and be more favorable to the status quo; also, given that the main results of the paper apply to all binary-choice scenarios, it would be interesting to analyze how different explanations of turnout that may apply to small and large numbers would suggest different systems for referenda and for decision-making in committees (where personal interaction is possible and strategic coalition behaviour may occur). All these questions, as well as possible justifications for the choices implied by the main voting systems, are open to further research.

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Table 1: Examples of referendum systems

Type	Example	1. Majority	2. Voting Threshold	3. Majority Threshold
S	Estonia, Ireland, Liechtenstein, Slovenia (Act), Switzerland	$\beta > \frac{1}{2}$	-	-
S	Namibia	$\beta > \frac{2}{3}$		
MT	Germany (Land), Hungary	$\beta > \frac{1}{2}$	-	$\beta \frac{n'}{n} > \frac{1}{4}$
MT	Denmark (Act)	$\beta > \frac{1}{2}$	-	$\beta \frac{n'}{n} > \frac{3}{10}$
MT	Denmark (Const)	$\beta > \frac{1}{2}$	-	$\beta \frac{n'}{n} > \frac{2}{5}$
MT	Belarus	$(\beta > \frac{1}{2})$	-	$\beta \frac{n'}{n} > \frac{1}{2}$
VT	Italy, Portugal, Romania, Slovenia (Const), Slovakia	$\beta > \frac{1}{2}$	$\frac{n'}{n} > \frac{1}{2}$	-

Source: Constitutions of the countries

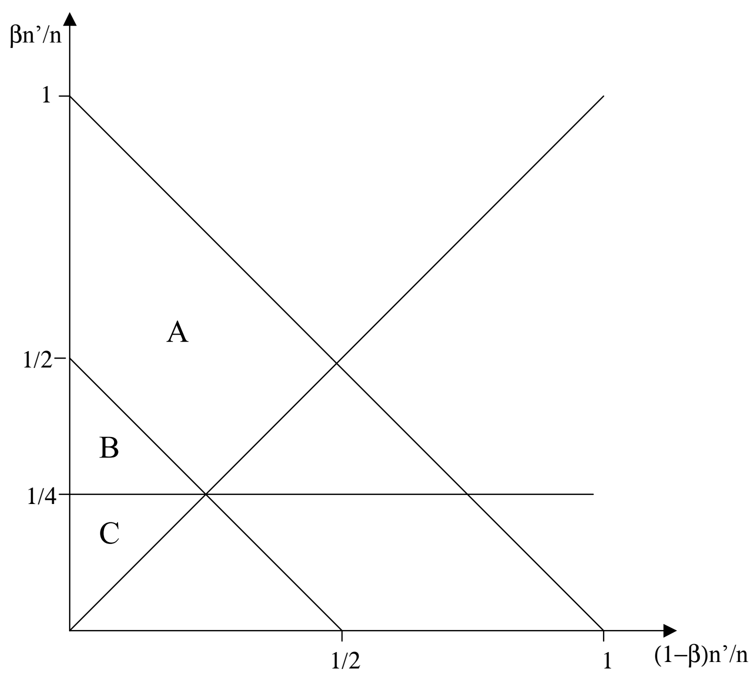

Figure 1: All points in regions A, B and C satisfy the Majority condition; however, no point in C satisfies the Majority Threshold of $\frac{1}{4}$ and no point in either B or C satisfies the Voting Threshold of $\frac{1}{2}$.

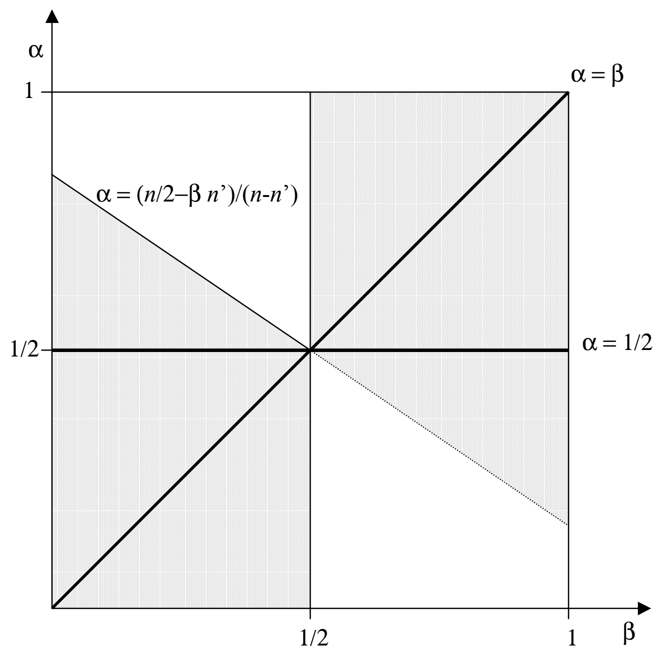

Figure 2: For the $S(\frac{1}{2})$ referendum, α must be in the shaded region for all possible polities. Two functions that would satisfy the requirement are $\alpha = \frac{1}{2}$ (compatible with the idea of indifference of abstainers with respect to the outcome) and $\alpha = \beta$ (compatible with the idea that the voters and the abstainers share the same preferences).

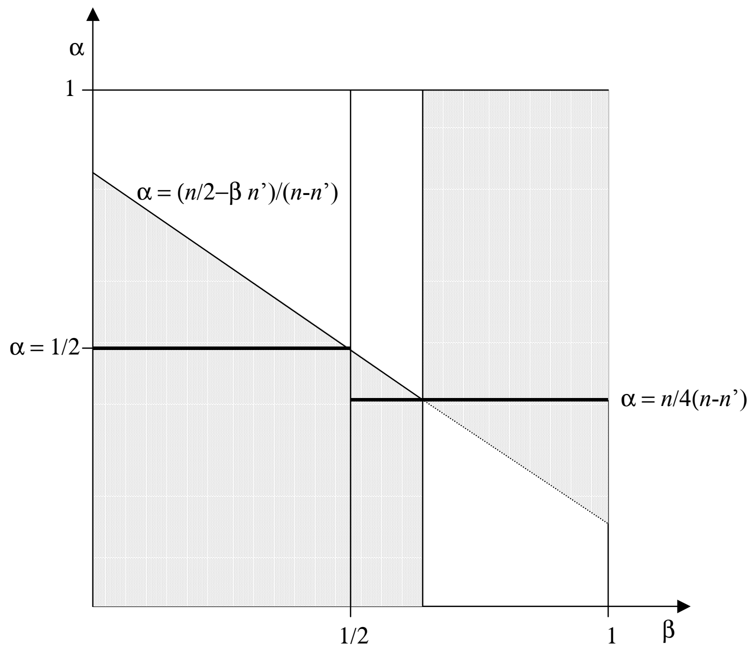

Figure 3: This is the case for $MT(1/4)$ where $\frac{n}{4(n-n')} \leq \frac{1}{2}$ or, equivalently, $n' \leq \frac{n}{2}$. Yes wins only if β is above $\frac{n}{4n'}$, the value that corresponds to the intersection of $\left(\frac{n}{2} - \beta n'\right)/(n - n')$ and $\frac{n}{4(n-n')}$. For the case where $n' > \frac{n}{2}$, we can simply use $\alpha = \frac{1}{2}$ and go back to Figure 2.

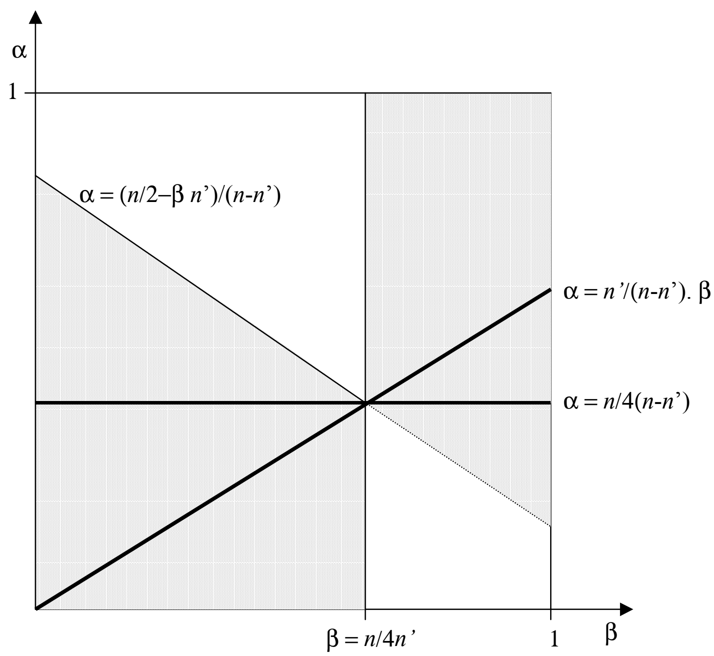

Figure 4: This is again the case for $MT(1/4)$ where $\frac{n}{4(n-n')} \leq \frac{1}{2}$ or, equivalently, $n' \leq \frac{n}{2}$. Yes wins only if β is above $\frac{n}{4n'}$, the value that corresponds to the intersection of $\left(\frac{n}{2} - \beta n'\right)/(n-n')$ with both $\frac{n}{4(n-n')}$ and $\frac{n'}{n-n'}.\beta$. For the case where $n' > \frac{n}{2}$, we can simply use $\alpha = \frac{1}{2}$ and go back to Figure 2.

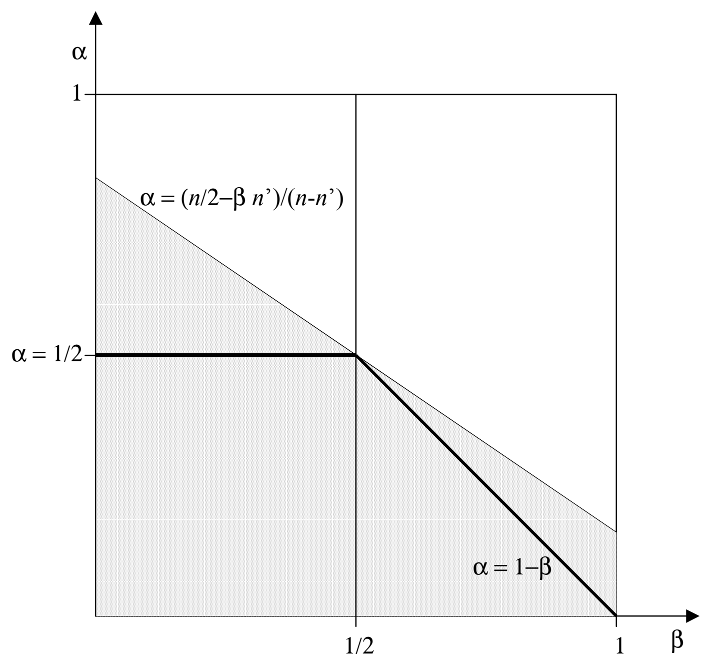

Figure 5: This is a depiction of $VT(1/2)$, when $n' \leq n/2$. In this case, $(n/2 - \beta n')/(n - n') \geq 1 - \beta$ for $\beta > 1/2$ and the rule selects No for all β.

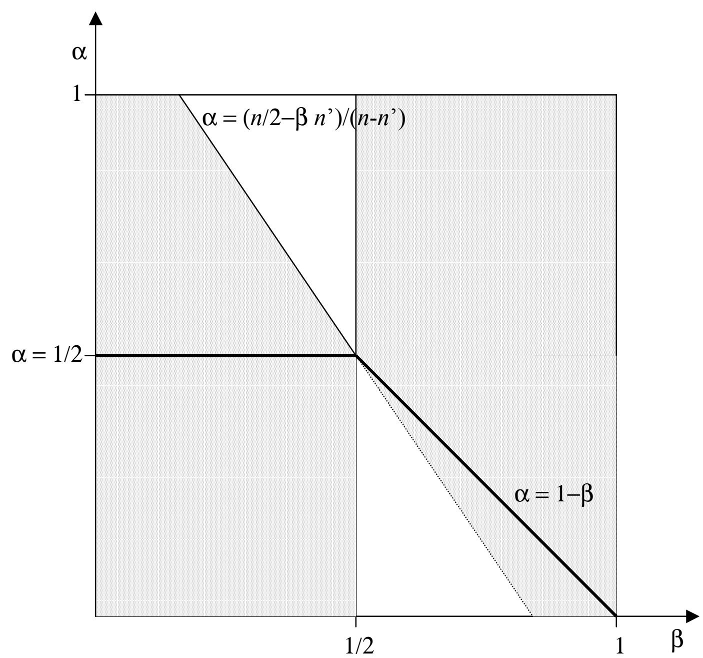

Figure 6: Here, we have $n' > n/2$. In this case, $(n/2 - \beta n')/(n - n') < 1 - \beta$ and the rule selects Yes for $\beta > 1/2$. Having $\alpha = 1/2$ if $\beta \leq 1/2$ and $\alpha = 1 - \beta$ if $\beta > 1/2$ accurately represents the decisions in both cases.